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New classes of solutions for the Schwarzian Korteweg–de Vries equation in (2+1) dimensions*

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Abstract

We have obtained new classes of solutions for the (2+1)-dimensional Schwarzian Korteweg–de Vries equation by considering several types of reductions of a system equivalent to this equation. The first analysis is done by studying the nonclassical reductions of the system. Further reductions are attained by means of other types of symmetry reductions or by *ansatz*-based reductions. Most of the new classes of solutions depend on Jacobian elliptic functions and solutions of a Riemann wave equation, including the cnoidal waves solutions. The new classes of solutions can display several types of coherent structures and can exhibit the overturning or intertwining phenomena, according to the suitable selection of the functions these solutions depend on.

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1. Introduction

The Korteweg–de Vries equation (KdV) is an integrable equation which possess a wealth of interesting properties and has been the origin of many others integrable equations [1]. Some of these extensions are based on a result of S Lie, who proved that the Schwarzian of a function f is the unique elementary function of the derivatives Df of f, excluding f itself, which is invariant under Möbius transformations. The fact that the KdV equation possesses infinitely many symmetries has been related with the invariance under Möbius transformations of the Schwarzian Korteweg–de Vries (SKdV) equation

$$-\frac{\phi_t}{\phi_x} = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{1}$$

where the right-hand side is the Schwarzian derivative of ϕ [11]. This equation was introduced in [12, 20] and has been studied in [4, 10, 22].

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By the other side, the investigation of the Painlevé property for some other well-known partial differential equations (Burgers, mKdV, Bousinesq, KP equations) reveals that the consideration of the 'singular manifold' does also lead to a formulation of these equations in terms of the Schwarzian derivative. Many interesting results concerning these integrable models (as the existence of infinitely many non-local or local symmetries) can be derived from the conformal invariance of their Schwarz forms.

Since most of former models are formulated for one spatial coordinate, many studies have been made to extend the models, by considering more spatial variables and conserving their integrable character. The typical way, but not the unique, to do this is to extend the Lax pair of the basic integrable equation to more spatial variables. The new corresponding system usually has solutions that have their counterpart in the basic equation but may also have a great variety of new solutions. Many new properties can be seen to be exhibited by such solutions which are not displayed by the corresponding basic (1+1)-dimensional system.

One of the first (2+1)-dimensional integrable systems that have been studied is the 'breaking soliton' equation, also named the Calogero–Bogoyavlenskii–Schiff (CBS) equation,

$$u_{xt} + u_x u_{xz} + \tfrac{1}{2} u_z u_{xx} + \tfrac{1}{4} u_{xxxz} = 0, \tag{2}$$

which can directly be obtained from the KdV equation by the method of extending the corresponding Lax pair. Bogoyavlenskii [3] obtained, by means of a Miura transformation, the following modified version of equation (2):

$$0 = 4u_t + u_{xxxy} - 4u^2 u_y - 4u_x \partial_x^{-1}(u u_y), \tag{3}$$

where $V = \partial_x^{-1} f = \int f \, \mathrm{d}x$ stands for a non-local variable such that $V_x = f$ [18]. Bogoyavlenskii also obtained the corresponding Lax pair and a non-isospectral condition for the spectral parameter. Equation (3) was also derived by Kudryashov and Pickering [13] as a member of a (2+1)-dimensional Schwarzian breaking soliton hierarchy. Estévez and Prada have shown that equation (3) can also be considered as a generalization to 2+1 dimensions of the sine-Gordon equation [6]. Another interesting equation that is also related to KdV and mKdV, and the Schwarzian operator, is the $(2+1)$-dimensional version of the Krichever–Novikov equation. This equation has been studied by Dorfman and Nijhoff [5] in the context of Hamiltonian structures. These authors proved its invariance under the Möbius group.

Higher dimensional integrable equations are not usually unique, in the sense that there exist several equations that reduce to a given one under dimensional reductions. The Schwarzian equation (1) has also been the basis for several generalizations and extensions. One of them, obtained by Toda and You [19] by extending the corresponding Lax pairs of equation (1), is the following:

$$W_t + \frac{1}{4} W_{xxz} - \frac{W_x W_{xz}}{2W} - \frac{W_{xx} W_z}{4W} + \frac{W_x^2 W_z}{2W^2} - \frac{W_x}{8} \left(\partial_x^{-1} \left(\frac{W_x^2}{W^2} \right) \right)_z = 0. \tag{4}$$

In terms of a potential ϕ such that $W = \phi_x$, the first member of (4) can be written as the derivative, with respect x, of

$$\phi_t + \frac{\phi_x}{4} \left[\left(\frac{\phi_{xx}}{\phi_x} \right) - \frac{1}{2} \partial_x^{-1} \left(\frac{\phi_{xx}^2}{\phi_x^2} \right) \right]_z . \tag{5}$$

By integrating (4) with respect to x, and assuming that the integration constant (an arbitrary function of (z, t)) is null, equation (4) is converted into

$$-4 \frac{\phi_t}{\phi_x} = \left[\left(\frac{\phi_{xx}}{\phi_x} \right) - \frac{1}{2} \partial_x^{-1} \left(\frac{\phi_{xx}}{\phi_x} \right)^2 \right]_z . \tag{6}$$

The form of equation (6) implies that if ϕ is a solution of (6) then $c(z)\phi$ is also a solution of the equation, for any arbitrary smooth function $c(z)$.

Supposing null the integration constant is not restrictive because, if W satisfies (4), we can always choose a potential ϕ for which (5) is null. It should be observed that the right-hand side of equation (6) is the (2+1)-dimensional Schwarzian derivative of ϕ. Equation (4) is named in [19] as the (2+1)-dimensional Schwarzian Korteweg–de Vries equation ((2+1) SKdV equation). Equation (4) passes the integrability Painlevé test, in the sense of the Weiss–Tabor–Carnevale method, and is invariant under Möbius transformation [19]. As a consequence, if W is a non-null solution of (4), then $1/W$ is also a solution of (4).

By setting

$$u = \frac{\phi_x}{\phi}, \qquad v = \frac{\phi_t}{\phi}, \tag{7}$$

equation (6) is converted into the system

$$\begin{aligned} &4u^2 v_x - 4uu_x v + u^2 u_{xxz} - uu_{xx}u_z - 3uu_x u_{xz} + 3u_x^2 u_z - u^4 u_z = 0, \\ &u_t - v_x = 0. \end{aligned} \tag{8}$$

This system is related, through a Miura transformation, to the Ablowitz–Kaup–Newell–Segur (AKNS) equation [16].

Some explicit solutions of (4) have been found by analysing system (8) [7, 15]: travelling waves, kinks, solitons and multi-solitons, etc. In [16], we obtained several classes of solutions that exhibit overturning and intertwining phenomena. One of these classes of solutions can be transformed into the solutions found by Bogoyavlenskii [3] for equation (2). These solutions of (4) depend on functions which are solutions of the well-known Riemann wave equation $\Phi_t + \Phi\Phi_z = 0$, which is also called the inviscid Burgers equation and has a great interest in hydrodynamics and traffic flow. Solutions of the Riemann wave equation have overturning (whiplash) phenomena in the wave front for all nonconstant solutions, which become multiple-valued [17]. There is no contradiction between the fact that these solutions of (4) are multi-valued functions and the fact that solutions of (4) are single-valued around all non-characteristic hypersurfaces, because solutions do branch at characteristic hypersurfaces only. Some aspects of this class of solutions have been studied by Whitham [21]. Although overturning solutions do not frequently appear in the literature, they have been found for some nonlinear coupled systems and the breaking soliton equation [8, 9].

The role of the integration constants is important for the relationship between equation (4) and system (8). These constants do not only appear for the term $\partial_x^{-1}\left(\frac{W_x^2}{W^2}\right)$: they are present in the fact that (5) is null and in the determination of ϕ from u and v. This has led us to study (4) by introducing the non-local variable $V = \partial_x^{-1}\left(\frac{W_x^2}{W^2}\right)$; the resulting equivalent system, of partial differential equations without non-local terms, can be written as

$$\begin{aligned} &W_t + \frac{1}{4}W_{xxz} - \frac{W_x W_{xz}}{2W} - \frac{W_{xx}W_z}{4W} + \frac{W_x^2 W_z}{2W^2} - \frac{W_x}{8}V_z = 0, \\ &W^2 V_x = W_x^2. \end{aligned} \tag{9}$$

Although systems (8) and (9) are clearly related, they are not completely equivalent because of the integration processes we have mentioned. However, it is easy to check that a given solution of any of these systems leads to a family of solutions of the other system, depending on some functions of t and/or z.

We focus our attention in obtaining new classes of solutions of (4) by studying the non-classical symmetries of system (9). Although a complete analysis of the nonclassical symmetries of a system can rarely be done, it turns out that the determining equations for the non-classical symmetries of (9) are less involved than those corresponding to (8). This let us to find reductions for (4) that do not have their counterpart for (8). As a consequence, we can widen the class of known solutions. The solutions that can be obtained through this method include the solutions found in the papers we have mentioned above. In particular, we have found several classes of solutions that can be expressed in terms of the Jacobi elliptic functions; some of these solutions are related with the well-known *cnoidal waves* for the Korteweg–de Vries equation [1].

2. First reductions: nonclassical symmetries

In order to obtain new solutions of equation (4), we have applied the nonclassical method of reduction to system (9). We consider a one-parameter Lie group of infinitesimal transformations in (x, z, t, W, V) and the associated Lie algebra of infinitesimal symmetries of the form

$$G = X\partial_x + Z\partial_z + T\partial_t + \overline{W}\partial_W + \overline{V}\partial_V, \tag{10}$$

where $X, Z, T, \overline{W}$ and $\overline{V}$ depend on (x, z, t, W, V). We require that (10) leaves invariant system (9) and the invariant surface conditions

$$XW_x + ZW_z + TW_t - \overline{W} = 0, \qquad XV_x + ZV_z + TV_t - \overline{V} = 0. \tag{11}$$

This yields to an overdetermined nonlinear system of equations for the infinitesimals $X, Z, T, \overline{W}$ and $\overline{V}$. There are several cases to consider.

Case 1. $T \neq 0$.

In this case we can set $T = 1$, without loss of generality, and we obtain the infinitesimals

$$\begin{aligned} &X = \alpha(t)x + \beta(t), && Z = \eta(z,t), \quad T = 1, \\ &\overline{W} = \gamma(z)\eta(z,t)w, && \overline{V} = -\alpha v - 8z(\beta' + 2\alpha\beta) - 8\beta\eta + \nu(t), \end{aligned} \tag{12}$$

where $\beta(t)$, $\gamma(z)$ and $\nu(t)$ are arbitrary functions and $\alpha(t)$ and $\eta(z, t)$ must satisfy the following equations:

$$\begin{aligned} &\alpha' + 2\alpha^2 + \alpha\eta_z = 0, \\ &\eta_t + \eta\eta_z + 2\alpha\eta = 0. \end{aligned} \tag{13}$$

These nonclassical symmetries of system (9) are different from the nonclassical symmetries of system (8) that have been analysed in [16]. The infinitesimals that correspond to the independent variables must, obviously, be the same; however, the infinitesimals that correspond to dependent variables are different; they depend on three arbitrary functions while the infinitesimals in [16] do only depend on a single arbitrary function.

We can distinguish two subcases.

Subcase 1.1. If $\alpha \neq 0$ then system (13) leads to a classical symmetry and the corresponding reductions are related to a reduction that has already been considered in [15].

Subcase 1.2. If $\alpha = 0$ then system (13) is reduced to the Riemann equation $\eta_t + \eta\eta_z = 0$. In this case, by solving the corresponding characteristic equations, we obtain the reductions

$$W = c(z)f(w, \eta), \qquad V = g(w, \eta) + 8za'(t) + \mathrm{d}(t), \tag{14}$$

where $w = x + a(t)$ and $\eta = \eta(z, t)$ is a solution of the Riemann equation $\eta_t + \eta\eta_z = 0$. By using (14), system (9) reduces to

$$\begin{aligned} &4f_\eta f_w^2 - 2f(2f_w f_{w\eta} + f_\eta f_{ww}) - f^2(8\eta f_\eta + g_\eta f_w - 2f_{ww\eta}) = 0, \\ &f_w^2 - f^2 g_w = 0. \end{aligned} \tag{15}$$

Case 2. $T = 0$, $Z \neq 0$. Without loss of generality, we can assume that $Z = 1$. Two subcases appear for the coefficients of (10).

Subcase 2.1. If X is not constant, then we obtain the infinitesimals

$$\begin{aligned} X &= \frac{x+\alpha(t)}{c_1 - 2z}, \qquad Z = 1, \quad T = 0, \\ \overline{W} &= b(z)W, \qquad \overline{V} = \frac{-V - \beta(t) - 8z\alpha'}{c_1 - 2z}, \end{aligned} \tag{16}$$

which lead to solutions of the form

$$\begin{aligned} &W = c(z)f(w,t); \\ &V = \sqrt{z+c_1}\,g(w,t) + \beta(t) + 8\alpha'(z+c_1), \end{aligned} \tag{17}$$

where $w = \sqrt{z+c_1}(x+\alpha)$.

The corresponding reduced system is

$$\begin{aligned} &4wf_w^3 - 2ff_w(2f_w + 3wf_{ww}) + f^2(16f_t - f_w(g + wg_w) + 4f_{ww} + 2wf_{www}) = 0, \\ &f_w^2 - f^2 g_w = 0. \end{aligned} \tag{18}$$

If $f_w \neq 0$, system (18) is equivalent to the equation

$$\begin{aligned} 3wf_w^5 - 3ff_w^3(f_w + 2wf_{ww}) + f^2f_w^2(5f_{ww} + 3wf_{www}) + f^3(f_{ww}(8f_t + 2f_{ww} + ff_{www}) \\ - f_w(8f_{wt} + 3f_{www} + wf_{wwww})) = 0. \end{aligned} \tag{19}$$

Subcase 2.2. If X is constant, then we obtain the infinitesimals

$$X = c_1, \qquad Z = 1, T = 0, \qquad \overline{W} = c(z)W, \qquad \overline{V} = a(t).$$

The corresponding reductions of (9) have the form

$$\begin{aligned} &y = x + c_2 z, \\ &W = c(z)f(y,t), \\ &V = za(t) + g(y,t), \end{aligned} \tag{20}$$

where f and g must satisfy the system

$$\begin{aligned} &c_2\left(4f_y^3 - ff_y(fg_y + 6f_{yy}) + 2f^2 f_{yyy}\right) + f^2(8f_t - af_y) = 0, \\ &f_y^2 - f^2 g_y = 0. \end{aligned} \tag{21}$$

Case 3. If $T = 0$, $Z = 0$, $X \neq 0$. In this case, without loss of generality, we can assume that $X = 1$. The corresponding infinitesimals are given by

$$X = 1, \qquad Z = 0, \qquad T = 0, \qquad \overline{W} = \rho W, \qquad \overline{V} = \rho^2, \tag{22}$$

where $\rho = \rho(z,t)$ is a solution of the Riemann equation $4\rho_t - \rho^2\rho_z = 0$. The infinitesimals (22) give us solutions of (9) of the form

$$W = \exp(\rho x + a(z,t)), \qquad V = \rho^2 x + b(z,t), \tag{23}$$

where $a(z,t)$ and $b(z,t)$ are related to the reduced equation

$$8a_t - \rho b_z = 0. \tag{24}$$

3. Further reductions

The nonclassical symmetries of system (9), we have found in section 2, have led to several reduced systems with *only* two independent variables. However, these reduced systems are rather involved and, in order to obtain some solutions of them, several additional studies must be made. In some cases additional reductions can be obtained by studying the symmetries of these reduced systems. However, in other cases, the reductions we have been able to find are consequence of the setting of specific *ansatzs*. We will follow the subcases we have considered in section 2.

Case 1.2. For the analysis of system (15), we have followed two different ways, which lead to different classes of solutions.

1.2.1. First, we have made a symmetry analysis of system (15). This gives us a four-parametric Lie group generated by the vector fields

$$V_1 = \frac{\partial}{\partial w}, \qquad V_2 = \frac{\partial}{\partial g}, \qquad V_3 = f\frac{\partial}{\partial f}, \qquad V_4 = w\frac{\partial}{\partial w} - 2\eta\frac{\partial}{\partial \eta} - g\frac{\partial}{\partial g}.$$

The first two symmetry groups are trivial (translations). The invariant solutions that correspond to V_3 and V_4 are of the form

$$f = (\epsilon\eta)^{n/2}h(y), \qquad g = \sqrt{\epsilon\eta}k(y), \tag{25}$$

where $y = \sqrt{\epsilon\eta}w$, $\epsilon^2 = 1$ and $n \in \mathbb{R}$.

The corresponding reduced system of ordinary differential equations is

$$\begin{aligned} &-4y(h')^3 + 2hh'(2h' + 3yh'') + h^2(h'(k + yk') - 4h'' - 2yh''') + 8\epsilon h^2(nh + yh') = 0,\\ &(h')^2 - h^2k' = 0, \end{aligned} \tag{26}$$

which is equivalent to the equation

$$\begin{aligned} 3y(h')^5 - 3h(h')^3(h' + 2yh'') + h^2(h')^2(5h'' + 3yh''') + h^3(2(h'')^2 \\ + yh''h''' - h'(3h''' + yh'''')) + 4\epsilon h^3((1+n)(h')^2 - nhh'') = 0. \end{aligned} \tag{27}$$

Equation (27) admits the symmetry $h\frac{\partial}{\partial h}$. By setting $p = \frac{h'}{h}$, equation (27) is converted into

$$p^4 + yp^3p' + p'(2p' + yp'') - p(3p'' + yp''') + 4\epsilon(p^2 - np') = 0. \tag{28}$$

We are not able to obtain the general solution of (28). However, we have found some particular solutions of (28), by setting some specific *ansatzs*, that lead to solutions of (27). In the following table we now list some solutions of equation (27), depending on the constants n and ϵ:

	n	ϵ	h
1.2.1.1	Arbitrary	Arbitrary	$h = c_1 y^{-n}$
1.2.1.2	Arbitrary	-1	$h = \exp(\pm 2y + c_2)$
1.2.1.3	0	1	$h = \tanh^{\pm 2}(y)$
1.2.1.4	0	-1	$h = \tan^{\pm 2}(y)$
1.2.1.5	∓ 1	1	$h = (c_1 \sin(y) + c_2 \cos(y))^{\pm 2}$
1.2.1.6	∓ 1	-1	$h = (c_1 \exp(y) + c_2 \exp(-y))^{\pm 2}$
1.2.1.7	∓ 2	1	$h = (1 - y \coth(y))^{\pm 2}$
1.2.1.8	∓ 2	-1	$h = (1 - y \cot(y))^{\pm 2}$
1.2.1.9	∓ 2	1	$h = (1 - y \tanh(y))^{\pm 2}$
1.2.1.10	∓ 2	-1	$h = (1 + y \tan(y))^{\pm 2}$
1.2.1.11	∓ 2	Arbitrary	$h = k_1 (y + k_2)^{-n}$

Former solutions of equation (27) lead to corresponding solutions of (4). We only list some classes of solutions that do not appear in previous papers:

1.2.1.I. From (1.2.1.6), we obtain

$$\begin{aligned} &W = c(z)\rho^{\mp 1} \cdot (c_1 \exp(\rho \cdot (x + a(t))) + c_2 \exp(-\rho \cdot (x + a(t))))^{\pm 2}, \\ &\rho_t - \rho^2 \rho_z = 0, \end{aligned} \tag{29}$$

where c_1 and c_2 are arbitrary constants. If $c_1 = c_2 = \frac{1}{2}$, then (29) becomes

$$W = c(z)\rho^{\mp 1} \cosh^{\pm 2}(\rho \cdot (x + a(t))), \qquad \rho_t - \rho^2 \rho_z = 0. \tag{30}$$

This special class of solutions has already been considered in [16].

1.2.1.II. From (1.2.1.9), we get

$$W = c(z)\rho^{\pm 2} \cdot (1 - \rho \cdot (x + a(t)) \tanh(\rho \cdot (x + a(t))))^{\mp 2}, \qquad \rho_t + \rho^2 \rho_z = 0. \tag{31}$$

1.2.1.III. From (1.2.1.10)

$$W = c(z)\rho^{\pm 2} \cdot (1 + \rho \cdot (x + a(t)) \tan(\rho \cdot (x + a(t))))^{\mp 2}, \qquad \rho_t - \rho^2 \rho_z = 0. \tag{32}$$

1.2.1.IV. From (1.2.1.11)

$$W = c(z) \left(x + a(t) + \frac{k_2}{\sqrt{\eta}} \right)^{\pm 2}, \qquad \eta_t + \eta \eta_z = 0. \tag{33}$$

1.2.2. By the other side, the analysis of the way the variables (w, η, f, g) are involved in (15) leads us to search solutions of the form

$$\begin{aligned} &f = h(y), \\ &g = c_1 w + k(y) + 8(-\eta b'(\eta) + b(\eta)), \end{aligned} \tag{34}$$

where $y = w + b'(\eta)$, c_1 is an arbitrary constant and $b(\eta)$ is an arbitrary smooth function. Then, system (15) is reduced to

$$\begin{aligned} &4(h')^3 - hh'(hk' + 6h'') + 2h^2 h''' = 0, \\ &(h')^2 - h^2(c_1 + k') = 0. \end{aligned} \tag{35}$$

System (35) is equivalent to the autonomous equation

$$3(h')^3 - 6hh'h'' + h^2(c_1 h' + 2h''') = 0. \tag{36}$$

It must be observed that if h is a solution of (36), then $1/h$ is also a solution of (36). Equation (36) admits the symmetries

$$Z_1 = \frac{\partial}{\partial y}, \qquad Z_2 = \frac{h}{2}\frac{\partial}{\partial h}. \tag{37}$$

By using these symmetries, we can reduce equation (36) to a complicated first-order ordinary differential equation, whose solutions can be given in terms of elliptic functions. These last solutions give us solutions of equation (36) of the following forms.

1.2.2.I. Solutions that are product of elliptic functions:

$$h(y) = \mathrm{sn}^{n_1}(c_2 y, m)\,\mathrm{cn}^{n_2}(c_2 y, m)\,\mathrm{dn}^{n_3}(c_2 y, m), \qquad \begin{cases} n_1, n_2, n_3 \in \{-2, 0, 2\}, \\ n_1 + n_2 + n_3 \in \{-2, 0, 2\}, \end{cases} \tag{38}$$

where sn, cn and dn are some of the classical Jacobian elliptic functions, m is an arbitrary constant and c_1 and c_2 are related constants. Particular solutions of the form (38) are the following:

(a) $h(y) = \mathrm{sn}^{\pm 2}(c_2 y, m)$, with $c_1 = -4c_2^2(1+m)$.
(b) $h(y) = \mathrm{cn}^{\pm 2}(c_2 y, m)$, with $c_1 = 4c_2^2(2m-1)$.
(c) $h(y) = \mathrm{dn}^{\pm 2}(c_2 y, m)$, with $c_1 = -4c_2^2(m-2)$.

1.2.2.II. Solutions of the form

(a) $h(y) = (\sqrt{m}\,\mathrm{cn}(c_2 y, m) + \epsilon_1 \mathrm{dn}(c_2 y, m))^{\pm 2}$, with $c_1 = 2c_2^2(1+m)$, $\epsilon_1 \in \{-1, 1\}$,
(b) $h(y) = (\mathrm{cs}(c_2 y, m) + \epsilon_1 \mathrm{ds}(c_2 y, m))^{\pm 2}$, with $c_1 = 2c_2^2(1+m)$, $\epsilon_1 \in \{-1, 1\}$,
(c) $h(y) = (\mathrm{dc}(c_2 y, m) + \epsilon_1\sqrt{1-m}\,\mathrm{sc}(c_2 y, m))^{\pm 2}$, with $c_1 = 2c_2^2(1-2m)$, $\epsilon_1 \in \{-1, 1\}$,

where cs, ds, dc and sc are Jacobian elliptic functions.

1.2.2.III. When $c_1 = 0$, equation (36) does also admit solutions of the form $h(y) = (c_2 + c_3 y)^{\pm 2}$, where c_2 and c_3 are arbitrary constants.

Any of the solutions of equation (36) we have just found gives us a corresponding solution for equation (4) of the form

$$W(x, z, t) = c(z)h(x + a(t) + b'(\eta)), \tag{39}$$

where η is a solution of the Riemann equation $\eta_t + \eta\eta_z = 0$ and $a(t)$, $b(\eta)$ and $c(z)$ are arbitrary smooth functions.

It must be observed that, when h is of type (I), the corresponding class of solutions (39) does strictly contain the classes of solutions obtained in [16] for (4), i.e. those solutions correspond to special values of parameters m and n_i. Solutions that correspond to cases (II) and (III) do not appear in [16]. Some of the classes of solutions we have obtained are related to the well-known *cnoidal waves* for the Korteweg–de Vries equation [1].

Case 2.1. It can be checked that equation (19) admits the symmetries

$$X_1 = f\frac{\partial}{\partial f}, \qquad X_2 = w\frac{\partial}{\partial w} + 2t\frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial t}. \tag{40}$$

From these symmetries, we get the reduction

$$f = (t + c_2)^{c_3} h(y), \; y = \frac{w}{\sqrt{\epsilon(t + c_2)}},$$

where $\epsilon^2 = 1$. The corresponding reduced equation is

$$3y(h')^5 - 3h(h')^3(h' + 2yh'') + h^2(h')^2(5h'' + 3yh''') + h^3(2(h'')^2 + yh''h''' \\ - h'(3h''' + yh'''')) + 4\epsilon h^3((1 - 2c_3)(h')^2 + 2c_3 hh'') = 0. \tag{41}$$

It is clear that if we set $n = -2c_3$, then equation (41) coincides with equation (27); therefore, equation (41) admits the solutions we have given above (subcase 1.2.1). However, since the reductions are not the same, the corresponding solutions W of (4) have different forms. The new solutions have the following form:

$$W = c(z)(t + c_2)^{c_3} h \left(\sqrt{\frac{z + c_1}{\epsilon(t + c_2)}}(x + a(t)) \right), \tag{42}$$

where h is a solution of (41). Since $c(z)$ is an arbitrary function, these last solutions can be written as $W = \mathrm{d}(z)\rho^{-2c_3} h(\rho(x + a(t)))$, where $\rho = \sqrt{\frac{z+c_1}{\epsilon(t+c_2)}}$. It must be observed that ρ is a solution of the Riemann equation $\rho_t + \epsilon\rho^2\rho_z = 0$, where $\epsilon = \pm 1$. This explains that the reduced equations (27) and (41) are the same.

Some of the solutions we have found, by the former procedure, are the following:

$$W = c(z) \tanh^{\pm 2} \left(\sqrt{\frac{z + c_1}{t + c_2}}(x + a(t)) \right), \tag{43}$$

$$W = c(z) \tan^{\pm 2} \left(\sqrt{\frac{z + c_1}{-(t + c_2)}}(x + a(t)) \right), \tag{44}$$

$$W = c(z) \sqrt{\frac{z + c_1}{-(t + c_2)}} \cosh^{-2} \left(\sqrt{\frac{z + c_1}{-(t + c_2)}}(x + a(t)) \right), \tag{45}$$

$$W = c(z) \sqrt{\frac{-(t + c_2)}{z + c_1}} \cosh^{2} \left(\sqrt{\frac{z + c_1}{-(t + c_2)}}(x + a(t)) \right). \tag{46}$$

It is obvious that $\rho = \rho(z, t) = \sqrt{\frac{z+c_1}{\epsilon(t+c_2)}}$ is not defined in the whole plane (z, t); it is defined in $\mathcal{D} = \{(z, t) \in \mathbb{R}^2 : (z + c_1) \cdot \epsilon(t + c_2) > 0\}$. Therefore, solutions (43)–(46) are not defined in $\mathbb{R}^3$; they exploit at a finite time $t = -c_2$. This explains that the overturning phenomena do not appear for this class of solutions. Let us observe that (44) can be considered as a prolongation of (43) because $\tanh(\mathrm{i}\alpha) = -\mathrm{i}\tan(\alpha)$ for $\alpha \in \mathbb{R}$.

Case 2.2. System (21) can be reduced, by solving in g_y, to the equation

$$c_2\big(3f_y^3 - 6ff_yf_{yy} + 2f^2 f_{yyy}\big) + f^2(8f_t - af_y) = 0. \tag{47}$$

When $c_2 = 0$, the corresponding solutions of (4) have already been considered in [15]. If $c_2 \neq 0$, then (47) can be reduced to the equation

$$-24c_4h^3 - 9c_2h'^3 + 18c_2hh'h'' + h^2(8c_3wh' - 6c_2h''') = 0, \tag{48}$$

by choosing $a(t)$ of the form

$$a(t) = \frac{-8\alpha(t)}{c_3t + c_4} - \frac{8c_3 \int^t \alpha(\xi)(c_3\xi + c_4)^{-4/3}\, \mathrm{d}\xi}{3(c_3t + c_4)^{2/3}}$$

and

$$w = \frac{y}{(c_3t + c_4)^{1/3}} - \int^t \alpha(\xi)(c_3\xi + c_4)^{-4/3}\, \mathrm{d}\xi, \qquad f = (c_3t + c_4)^{c_4/c_3} h(w). \tag{49}$$

By using the change $b(w) = \frac{h'}{h}$, (48) is reduced to

$$b'' - \frac{4c_3}{3c_2}wb - \frac{b^3}{2} + \frac{4c_4}{c_2} = 0. \tag{50}$$

Equation (50) is of Painlevé II type. When $c_3 = 0$, equation (50) admits solutions that can be expressed in terms of elliptic functions.

It is clear that when $c_3 = c_4 = 0$, the substitutions (49) are meaningless; however, it can be checked that, when $a = 0$, equation (47) admits solutions of the form $f(y, t) = h(y)$, where h is a solution of the equation

$$3(h')^3 - 6hh'h'' + 2h^2h''' = 0. \tag{51}$$

This last equation coincides with (48), when $c_3 = c_4 = 0$. Therefore, the solutions of (47) derived from (51) can be considered as stationary solutions. Some solutions of (51) are

$$h = k_1\,\mathrm{sn}^{\pm 2}(k_2w, -1), \qquad h = k_1\,\mathrm{cn}^{\pm 2}(k_2w, 1/2), \qquad h = k_1\,\mathrm{dn}^{\pm 2}(k_2w, 2), \tag{52}$$

where k_1 and k_2 are arbitrary constants.

Case 3. Let us consider the reduced equation (24): $8a_t - \rho b_z = 0$. If $a(z, t)$ is an arbitrary smooth function and $\rho(z, t)$ does not vanish, then $b(z, t)$ must have the form

$$b(z, t) = 8\int^z \frac{a_t(z, t)}{\rho(z, t)}\,\mathrm{d}z + C(t).$$

It can be checked that the corresponding solutions of (4) do already appear in [16]. Furthermore, if ρ is a non-null constant solution of the Riemann equation $4\rho_t - \rho^2\rho_z = 0$, then we obtain a solution that has been considered in [15].

4. Qualitative analysis of some solutions

In this section, we study some qualitative aspects of the classes of solutions of (4) we have found in section 3. We will only analyse some of the solutions that have not been considered before. Since many of these solutions depend on Jacobi elliptic functions, let us recall [2] that for some special values of the parameters these functions coincide with some trigonometrical or hyperbolic ones; e.g. $\mathrm{sn}(z, 0) = \sin z$, $\mathrm{sn}(z, 1) = \tanh z$, $\mathrm{cn}(z, 0) = \cos z$, $\mathrm{cn}(z, 1) = \cosh^{-1} z$, etc. However, for most of the values of the parameters these Jacobian functions are periodic.

A. We first consider the class of solutions of the form (39)

$$W(x, z, t) = c(z)h(x + a(t) + b'(\eta(z, t))). \tag{53}$$

Solutions of this form correspond to travelling waves. The role of $a(t)$ is to determine the path of the wave in the direction of the x axis. In (53), $\eta(z, t)$ is a solution of the Riemann equation $\eta_t + \eta\eta_z = 0$.

A.1. When η is constant, the function $c(z)$ modulates the wave along lines parallel to the z axis and (53) is a single-valued regular function. Among the functions h that lead to solutions of (4) the most interesting ones are given by (38) and are product of elliptic functions. We begin the qualitative analysis of solutions (53) with the case where h is a cnoidal function.

A.1.1. Let us suppose that h is of the form $h(\alpha) = \mathrm{cn}^2(\alpha, m)$, where m is a real parameter. For the same m, we can consider several functions $c(z)$ which have different ways of localization. In figure 1, we take $a(t) = t$, $b'(\eta) = 0$ and the functions $c(z)$ and $h(\alpha)$ given in the following table. These plots are taken for $t = 0$.

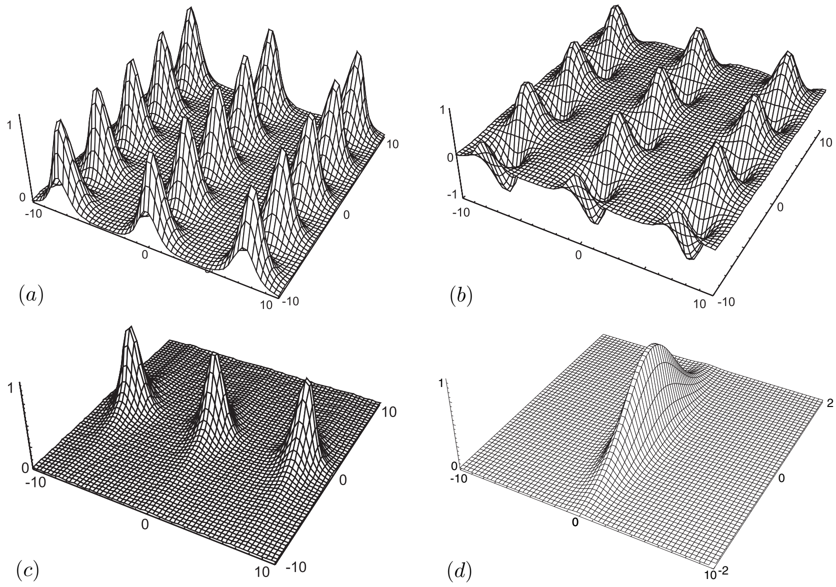

Figure 1. Some cnoidal solutions for the (2+1)-SKdV equation.

	(a)	(b)	(c)	(d)
c(z)	$\mathrm{sn}^2(z, 0.8)$	$\sin(z)$	$1/(z^2+1)$	e^{-z^2}
$h(\alpha)$	$\mathrm{cn}^2(\alpha, 0.99)$	$\mathrm{cn}^2(\alpha, 0.99)$	$\mathrm{cn}^2(\alpha, 0.99)$	$\mathrm{cn}^2(\alpha, 1)$

Figure 1(*a*) (resp. (*b*)) corresponds to a double periodic net of dromions (resp. dromions–antidromions), (*c*) displays a single periodic net of dromions and (*d*) shows an exponentially localized dromion or soliton.

A.1.2. Now, instead of the function $h(\alpha) = \mathrm{cn}^2(\alpha, m)$, we consider the elliptic functions $h(\alpha) = \mathrm{sn}^2(\alpha, m)$ and $h(\alpha) = \mathrm{dn}^2(\alpha, m)$, for several values of the parameter m. In figure 2, we plot the travelling wave solutions that correspond to the following data:

	(*a*)	(*b*)	(*c*)	(*d*)
c(z)	$1/(z^2+1)$	$-\,\mathrm{sn}^2(z, 0.99)$	$\mathrm{sn}^2(z, 0.5)$	$-\mathrm{sn}^2(z, 1)$
h(α)	$\mathrm{sn}^2(\alpha, 1)$	$\mathrm{sn}^2(\alpha, 0.99)$	$\mathrm{dn}^2(\alpha, 1)$	$\mathrm{dn}^2(\alpha, 1)$

The corresponding solutions of (4) display the following coherent structures: (*a*) and (*d*) show two types of kink–antikink structures, (*b*) shows a double periodic basin and (*c*) is a single periodic family of dromions.

A.2. Up to now, for the solutions we have considered in this section, we have assumed that $b'(\eta(z,t)) = 0$, where η is a solution of the Riemann equation $\eta_t + \eta\eta_x = 0$. If b' and η are non-constant functions then the overturning and/or intertwining phenomena, of the corresponding solutions W of (4), may appear [16]. This is because the solution of the Riemann equation $\eta_t + \eta\eta_z = 0$, with initial condition $\eta(z, 0) = k(z)$, is implicitly given by

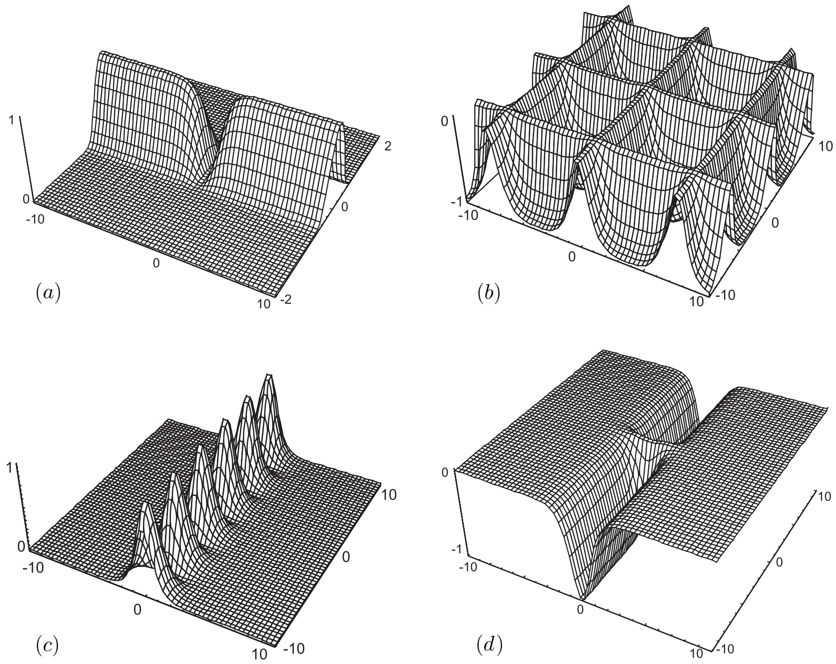

Figure 2. Some solutions for the (2+1)-SKdV equation related to several elliptic functions.

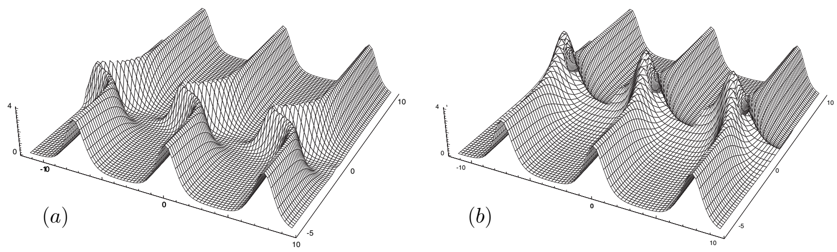

Figure 3. An overturning periodic solution for the (2+1)-SKdV equation.

$\eta = k(z - \eta t)$. Since the graph points of any solution η move, as t varies, parallel to the z axis with constant speed $k(z)$, the points that correspond to local maxima of $|k|$ move faster than the neighboring points; if k is not bounded, there are points in the graph that move with an arbitrary large speed.

The solutions of (4) that appear in [16] are particular cases of the classes we have obtained here. The new classes of solutions can also have the same overturning and/or intertwining phenomena that appear in [16] by considering non-constant solutions of the Riemann equation. However, for the new solutions, these phenomena could be periodically repeated. This happens, for instance, for $h(\alpha) = \mathrm{cn}^2(\alpha, 0.99)$ (that has been considered in A.1.1.) if we take $b'(\eta) = \eta$, where η is a solution of the Riemann equation $\eta_t + \eta\eta_z = 0$ satisfying the initial condition $\eta(z, 0) = \frac{1}{1+z^2}$. In figure 3, we display the corresponding solution for $t = 0$ and $t = 2$, where, to simplify the graphs, we have taken $c(z) = 4$, $a(t) = 0$.

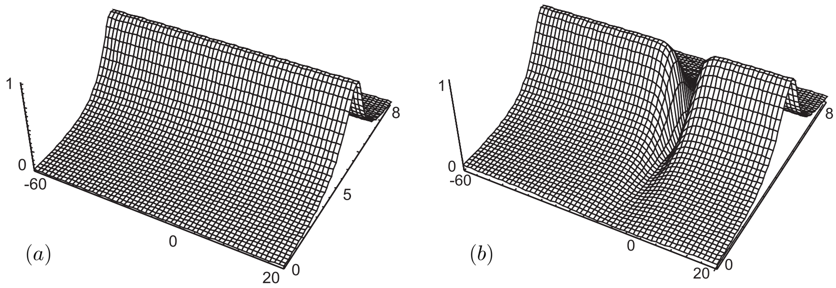

Figure 4. Time evolution to a bound state of kink–antikink type.

B. We now consider solutions of (4) that depend on unbounded solutions of a Riemann equation. In these cases, as we have mentioned before, there are points in the graph of η that move with an arbitrary large speed, as t grows, and some singularities appear.

B.1. Let us consider, for instance, the solutions of the form (43), with $c_1 = c_2 = 0$, that are not defined for $z < 0$ (resp. $z > 0$) if $t > 0$ (resp. $t < 0$). Clearly, $h(\alpha) = \tanh^2(\alpha)$ is an even function and, for every $\beta \in]0, 1[$, the equation $h(\alpha) = \beta$ has exactly two solutions that will be denoted by $\alpha_\beta > 0$ and $-\alpha_\beta < 0$. If we fix the pair (z, t), with $z > 0$ and $c(z) \neq 0$, the values x_1 and x_2 for which $W(x_i, z, t) = c(z)\beta$ are $x_1 = \frac{\alpha_\beta\sqrt{t}}{\sqrt{z}} - a(t)$ and $x_2 = \frac{-\alpha_\beta\sqrt{t}}{\sqrt{z}} - a(t)$. Therefore, $x_1 - x_2 = \frac{2\alpha_\beta\sqrt{t}}{\sqrt{z}}$. As a consequence, for a fixed $z > 0$ such that $c(z) \neq 0$, $x_1 - x_2$ is an increasing function of t that verifies $\lim_{t\to 0}(x_1 - x_2) = 0$ and $\lim_{t\to\infty}(x_1 - x_2) = +\infty$. This means that if $W(x, z, 0^+) = c(z)$ then, as t grows, the graph of W displays a growing opening that simultaneously is translated accordingly to $a(t)$. Therefore, an initially localized structure evolves to a bound state of type kink–antikink. These facts are represented in figure 4 for a function $c(z)$ that is localized around $z = 5$: $c(z) = 1/(1 + (z - 5)^2)$.

It should be observed that although $\rho(z, t) = \sqrt{\frac{z}{t}}$ is a non-null solution of the Riemann equation $\rho_t + \rho^2\rho_x = 0$, the solution we have just considered does not display the overturning or intertwining phenomena: this is because $\rho(z, t)$ is unbounded, as $t \to 0^+$.

B.2. We can also consider unbounded solutions of Riemann equations for solutions of (4) of the form (39) that have been considered before in this section. We take $h(\alpha) = \mathrm{sn}^2(\alpha, 1) = \tanh^2(\alpha)$, $c(z) = \tanh(z + 5) - \tanh(z - 5)$ and $\eta = \frac{z}{t}$. As before, for any $\beta \in]0, 1[$ and any fixed pair (z, t), with $z > 0$ and $c(z) \neq 0$, the values x_1 and x_2 for which $W(x_i, z, t) = c(z)\beta$ are $x_1 = \alpha_\beta - b'\left(\frac{z}{t}\right) - a(t)$ and $x_2 = -\alpha_\beta - b'\left(\frac{z}{t}\right) - a(t)$. Therefore, $x_1 - x_2 = 2\alpha_\beta$. As a consequence, for a fixed $z > 0$ such that $c(z) \neq 0$, $x_1 - x_2$ is independent of t. As t varies, the equation $x = b'(\frac{z}{t}) + a(t)$ gives us an one-parameter family of curves in the plane (x, z). Therefore, if $W(x, z, 0^+) = c(z)$ then, as t grows, the graph of W displays an opening of constant width localized around the curve $x = b'(\frac{z}{t}) + a(t)$. When $b'(u) = u$ and $a(t) = t$ that family of curves is the family of lines of equation $x = -\frac{z}{t} - t$; the slope of this line, with respect to the z axis, tends to 0 as $t \to \infty$ and its x-intercept is translated according to $a(t) = t$. These facts are represented in the graphs (*a*) and (*b*) of figure 5 for a function $c(z)$ that is localized around $z = 0$. In the graph (*c*) of this figure we display the time evolution of W for the case where $b'(u) = u^2$ and $a(t) = t$, with the same initial condition $W(x, z, 0^+) = c(z)$. In this last case the mentioned family of curves is a family of parabolas.

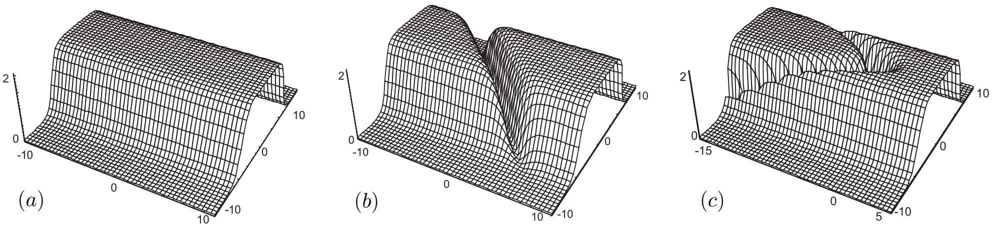

Figure 5. Two types of time evolution of structures with a moving opening.

5. Conclusions

For the (2+1) Schwarzian Korteweg–de Vries equation, we have found several new families of solutions that have not been considered before and another classes that contain, as particular cases, the known solutions of that equation. Most of the solutions functionally depend on solutions of Riemann wave equations; when these last solutions are not constant the solutions of the (2+1)-SKdV equation can display the overturning and/or intertwining phenomena. To obtain the first reductions we have used the non-classical method of symmetries; further reductions have been made by doing a symmetry analysis for any of the reduced equation or by setting some specific *ansatzs*. A qualitative analysis of some of these families has also been made. Several classes of coherent structures are displayed by some of the solutions: soliton, periodic families of dromions, periodic families of basins, kink-antikink structures, etc.

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